Final exam — Analysis (WPMA14004)

Thursday 18 June 2015, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (3 + 12 points)

(a) State the Axiom of Completeness.

(b) Assume that $A \subset \mathbb{R}$ is nonempty and bounded below. Consider the set

$$B = \{b \in \mathbb{R} \,:\, b \text{ is a lower bound for } A\}.$$

Prove that B is bounded above and $\inf A = \sup B$.

Problem 2 (5 + 3 + 7 points)

Consider the sequence (s_n) given by

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Prove the following statements:

(a) $s_{n+1} - s_n > 0$ for all $n \in \mathbb{N}$.

(b) $s_n \leq \dfrac{n}{n+1}$ for all $n \in \mathbb{N}$.

(c) $s = \lim s_n$ exists and $\frac{1}{2} \leq s \leq 1$.

Problem 3 (15 points)

Assume that (a_n) is a convergent sequence and $a = \lim a_n$. Consider the set

$$K = \{a_n \,:\, n \in \mathbb{N}\} \cup \{a\}.$$

Prove that K is compact.

Problem 4 (9 + 6 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that there exists a number $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$.

(a) Prove that f is uniformly continuous on $\mathbb{R}$.

(b) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = |f(x)|$. Prove that g is uniformly continuous on $\mathbb{R}$.

Problem 5 (4 + 4 + 7 points)

Consider the sequence $f_n(x) = (4x(1-x))^n$ on $A = [0, 1]$.

(a) Compute the pointwise limit of (f_n). Hint: first draw the graph of f_1.

(b) Does (f_n) converge uniformly on A?

(c) Let $|c| < 1$. Prove that $\displaystyle\sum_{n=1}^{\infty} c^n f_n(x)$ converges uniformly on A and compute the limit.

Problem 6 (3 + 6 + 3 + 3 points)

Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 1/p \text{ for some } p \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

(a) Show that $U(f, P) = 2$ for any partition P of $[0, 2]$.

(b) Prove that for all $\epsilon > 0$ there exists a partition P_ϵ of $[0, 2]$ such that $L(f, P_\epsilon) > 2 - 2\epsilon$. Hint: describe such a partition in words, rather than giving an explicit formula.

(c) Prove that f is Riemann-integrable on $[0, 2]$ and compute $\int_0^2 f$.

(d) Does there exist a function $F : [0, 2] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [0, 2]$?

End of test (90 points)

Solution of Problem 1 (3 + 12 points)

(a) Axiom of Completeness: every nonempty set of real numbers that is bounded above has a least upper bound.
(3 points)

(b) If $b \in B$, then $b \leq a$ for all $a \in A$. Therefore, every $a \in A$ serves as an upper bound for B, which shows that B is bounded above. Also note that B is nonempty since A is bounded below. By the Axiom of completeness $\sup B$ exists.
(4 points)

Since $\sup B$ is the *least* upper bound of B and every $a \in A$ is an upper bound for B it follows that $\sup B \leq a$ for all $a \in A$. This shows that $\sup B$ is a lower bound for A.
(4 points)

Now let ℓ be an arbitrary lower bound of A. Then $\ell \in B$ by definition of the set B. But then $\ell \leq \sup B$. This proves that $\sup B$ is also the *greatest* lower bound of A. We conclude that $\inf A = \sup B$.
(4 points)

Solution of Problem 2 (5 + 4 + 7 points)

(a) By definition we have

$$\begin{aligned} s_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \quad (n \text{ terms}), \\ s_{n+1} &= \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+1+n+1} \quad (n+1 \text{ terms}). \end{aligned}$$

Note that s_n and s_{n+1} have $n-1$ terms in common. Therefore, we obtain

$$s_{n+1} - s_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0.$$

(5 points)

(b) We have

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \leq \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} = \frac{n}{n+1}.$$

(3 points)

(c) From part (a) it follows that $s_n < s_{n+1}$ for all $n \in \mathbb{N}$, which means that (s_n) is increasing. From part (b) it follows that $s_n < 1$ for all $n \in \mathbb{N}$, which means that (s_n) is bounded above. Now apply the Monotone Convergence Theorem: every increasing sequence that is bounded above is convergent. We conclude that $s = \lim s_n$ exists.
(4 points)

Since $s_1 \leq s_n \leq 1$ for all $n \in \mathbb{N}$ and $s_1 = \frac{1}{2}$ it follows by the Order Limit Theorem that $\frac{1}{2} \leq s \leq 1$.
(3 points)

Solution of Problem 3 (15 points)

This problem has (at least) three different solutions.

Solution 1 (via open covers). To prove that K is compact we need to show that any open cover $\{O_i : i \in I\}$ for K has a finite subcover.
(2 points)

Since $a \in K$ there exists $i_0 \in I$ such that $a \in O_{i_0}$. Since O_{i_0} is open there exists a number $\epsilon > 0$ such that $V_\epsilon(a) \subset O_{i_0}$.
(3 points)

Because $\lim a_n = a$ there exists a number $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |a_n - a| < \epsilon \quad \Rightarrow \quad a_n \in V_\epsilon(a).$$

This proves that $a_n \in O_{i_0}$ for all $n \geq N$.
(5 points)

Since $a_1, \ldots, a_{N-1} \in K$ we can pick indices $i_1, \ldots, i_{N-1} \in I$ such that $a_k \in O_{i_k}$ for all $k = 1, \ldots, N-1$. We conclude that the collection $\{O_{i_0}, O_{i_1}, \ldots, O_{i_{N-1}}\}$ forms a finite subcover for K.
(5 points)

Solution 2 (via the Heine–Borel Theorem). To prove that K is compact we need to show that K is closed and bounded.
(2 points)

The sequence (a_n) is convergent and therefore bounded. Hence, there exists a number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. By the Order Limit Theorem it also follows that $|a| \leq M$. Therefore, $|x| \leq M$ for all $x \in K$, which shows that K is a bounded set.
(4 points)

To prove that K is closed we need to show that any Cauchy sequence K has a limit which is also contained in K. Let $(x_n) \subset K$ be Cauchy, then $x = \lim x_n$ exists. We need to prove that $x \in K$. The difficulty here is that (x_n) need not be a subsequence of (a_n)! We need to distinguish between three cases.

1. Assume that $x_n = a$ for infinitely many $n \in \mathbb{N}$. Then we can construct a subsequence (x_{n_k}) such that $x_{n_k} = a$ for all $k \in \mathbb{N}$. Hence, $x = \lim x_n = \lim x_{n_k} = a \in K$ and we are done.
   (3 points)
2. Assume that there exists $j \in \mathbb{N}$ such that $x_n = a_j$ for infinitely many $n \in \mathbb{N}$. Then we can construct a subsequence (x_{n_k}) such that $x_{n_k} = a_j$ for all $k \in \mathbb{N}$. Hence, $x = \lim x_n = \lim x_{n_k} = a_j \in K$ and we are done.
   (3 points)
3. If the previous cases do not apply, then infinitely many terms of (a_n) appear in the sequence (x_n). This means that there exists a subsequence (x_{n_k}) which, in turn, is a subsequence of (a_n). Hence, $x = \lim x_{n_k} = a \in K$ and we are done.
   (3 points)

Solution 3 (via the definition). To prove that K is compact we need to show that every sequence in K has a convergent subsequence with a limit in K.
(3 points)

Let $(x_n) \subset K$ be any sequence. To produce a convergent subsequence (x_{n_k}) with a limit in K we can proceed exactly as in Solution 2.
(4 points for each case)

Solution of Problem 4 (9 + 6 points)

(a) Let $x, y \in \mathbb{R}$ satisfy $x < y$. Recall the Mean Value Theorem: since f is differentiable on (x, y) and continuous on $[x, y]$ there exists a point $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y).$$

(3 points)

Taking absolute values gives

$$|f(x) - f(y)| = |f'(z)||x - y| \leq M|x - y|.$$

Note that this inequality continues to hold in the case $x \geq y$. Therefore, it holds for all $x, y \in \mathbb{R}$.
(3 points)

Take any number $\epsilon > 0$ and let $\delta = \epsilon/M$, then

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon.$$

Since $x, y \in \mathbb{R}$ are arbitrary, it follows that f is uniformly continuous on $\mathbb{R}$.
(3 points)

(b) The reverse triangle inequality implies that

$$\big|g(x) - g(y)\big| = \big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|.$$

(3 points)

Let $\epsilon > 0$ be arbitrary and take $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon \quad \Rightarrow \quad |g(x) - g(y)| < \epsilon,$$

which proves that also g is uniformly continuous on $\mathbb{R}$.
(3 points)

Solution of Problem 5 (4 + 4 + 7 points)

(a) Note that $f_n = f_1^n$ and f_1 is just a quadratic function. We have that $f_1(\frac{1}{2}) = 1$ and $0 \leq f_1(x) < 1$ for all $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Therefore,

$$f(x) = \lim f_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{if } x \neq \frac{1}{2} \end{cases}$$

(4 points)

(b) Since f_n is a polynomial it is a continuous function. If $f_n \to f$ uniformly on $[0, 1]$ then f would be continuous on $[0, 1]$. However, f has a jump discontinuity at $x = \frac{1}{2}$. We conclude that (f_n) does not converge uniformly to f.
(4 points)

(c) For $x \in [0, 1]$ we have

$$|c^n f_n(x)| = |c^n(4x(1 - x))^n| = |c|^n |4x(1 - x)|^n \leq |c|^n$$

Since $|c| < 1$ we can apply the Weierstrass M-test with $M_n = |c|^n$:

$$\sum_{n=1}^{\infty} M_n \text{ converges} \quad \Rightarrow \quad \sum_{n=1}^{\infty} c^n f_n(x) \text{ converges uniformly on } [0,1].$$

(4 points)

To compute the limit we use the geometric series with $r = 4cx(1-x)$:

$$\sum_{n=1}^{\infty} c^n f_n(x) = \sum_{n=1}^{\infty} (4cx(1-x))^n = \frac{1}{1-4cx(1-x)} - 1 = \frac{4cx(1-x)}{1-4cx(1-x)}.$$

(3 points)

Solution of Problem 6 (3 + 6 + 3 + 3 points)

(a) Let P be any partition of $[0,2]$. Every subinterval $[x_{k-1}, x_k]$ of P contains a point different from points of the form $1/p$ with $p \in \mathbb{N}$. Therefore,

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1$$

for all $k = 1, \ldots, n$, where n is the number of intervals in P. This shows that

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 2 - 0 = 2.$$

(3 points)

(b) Let $\epsilon > 0$ be arbitrary. Take $[0, \epsilon]$ as the first subinterval of P_ϵ. In that interval the infimum of f is 0. Note that the number of points of the form $1/p$ with $p \in \mathbb{N}$ outside $[0, \epsilon]$ is finite: there are at most $1/\epsilon$ many of such points. Around those points we take intervals of length less than ϵ^2. In these intervals the infimum of f is also 0. In all other intervals the infimum is 1.

(3 points)

This means that the total length of all intervals in which the infimum of f is 0 is at most $\epsilon + \epsilon^2 \cdot (1/\epsilon) = 2\epsilon$. Therefore,

$$\begin{aligned} L(f, P_\epsilon) &= \text{total length of all intervals in which } \inf f = 1 \\ &= 2 - \text{total length of all intervals in which } \inf f = 0 \\ &> 2 - 2\epsilon \end{aligned}$$

(3 points)

(c) Let $\epsilon > 0$ be arbitrary and take a partition P_ϵ of $[0,2]$ such that $L(f, P_\epsilon) > 2 - 2\epsilon$. By part (a) it follows that $U(f, P_\epsilon) = 2$. Therefore,

$$U(f, P_\epsilon) - L(f, P_\epsilon) < 2 - (2 - 2\epsilon) = 2\epsilon,$$

which proves that f is integrable on $[0,2]$.

(1 point)

By definition of the Riemann integral and part (a) we have

$$\int_0^2 f = U(f) = \inf\big\{U(f,P) : P \text{ is a partition of } [0,2]\big\} = 2.$$

(2 points)

(d) If there exists a function $F : [0, 2] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [0, 2]$ then Darboux's Theorem implies that f attains all values between 0 and 1, which is clearly not the case. Therefore, we conclude that no such F exists.
(3 points)